20th Oct. 2020 @ KEK theory seminar

Manifestly Covariant Theory of Stochastic Inflation

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w/ L. Pinol, S. Renaux-Petel 2008.07497

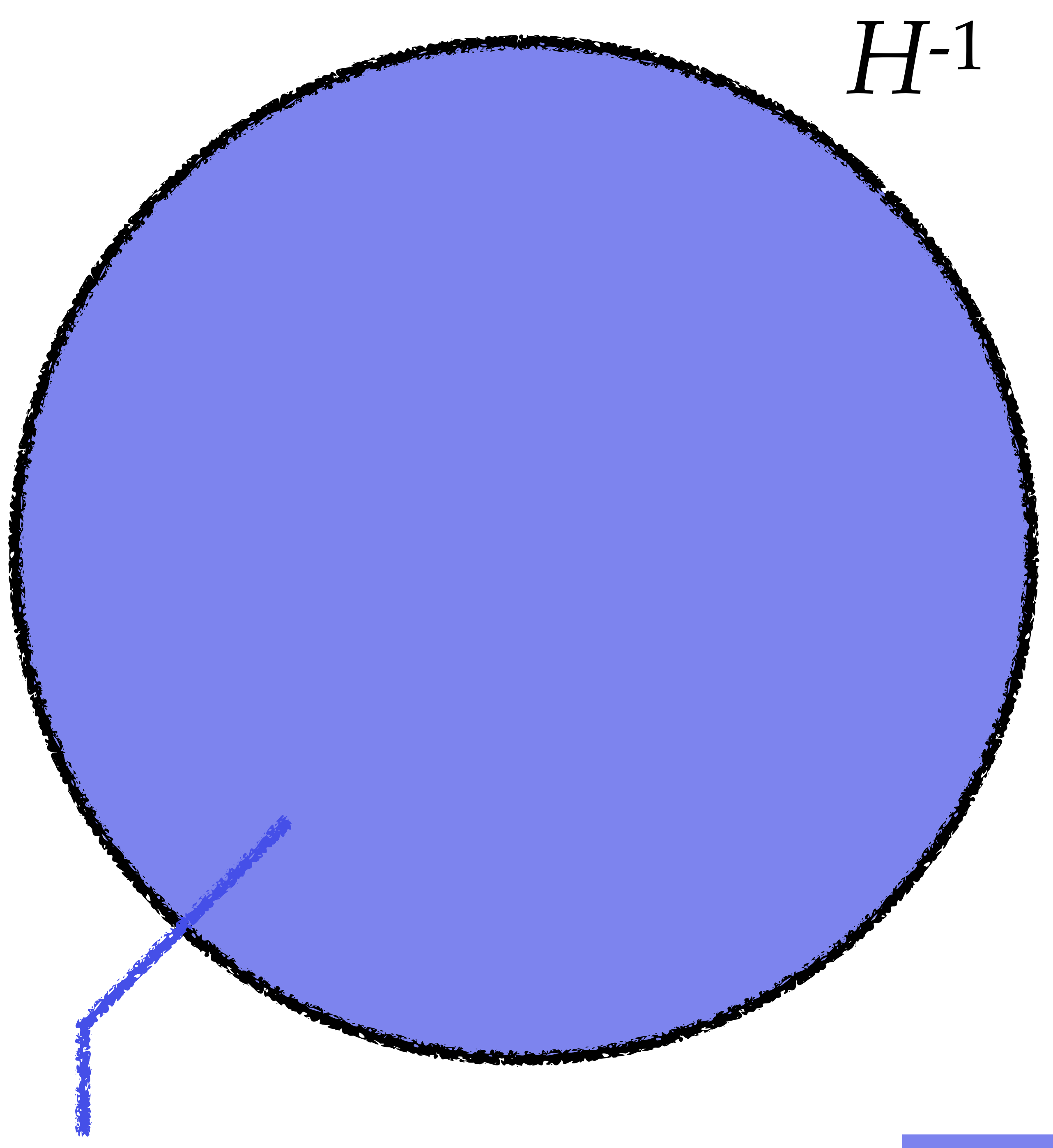
ref. Pinol, Renaux-Petel, Tada CQG **36** (2019) 07LT01
Renaux-Petel, Tada, Vennin in prep.

Contents

- Stochastic approach to inflation: *What? & How useful?*
- Ambiguity existed in formulation: *Stochastic Anomaly*
- Mathematically well-defined *covariant* formulation

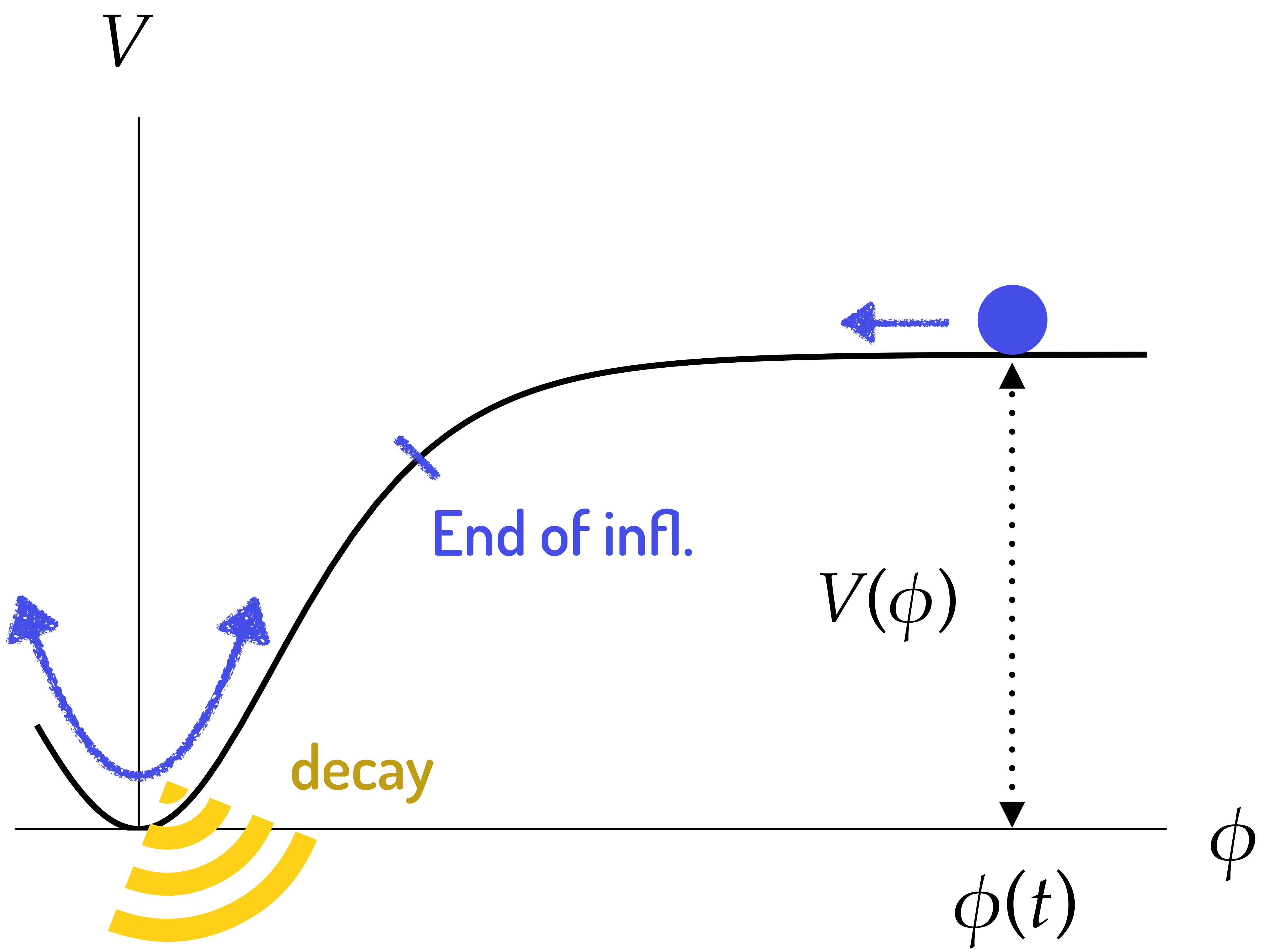
Inflation

pre-Big-Bang accelerated expansion



"Dark Energy": $\rho_{\rm const} = V(\phi_{\rm const})$

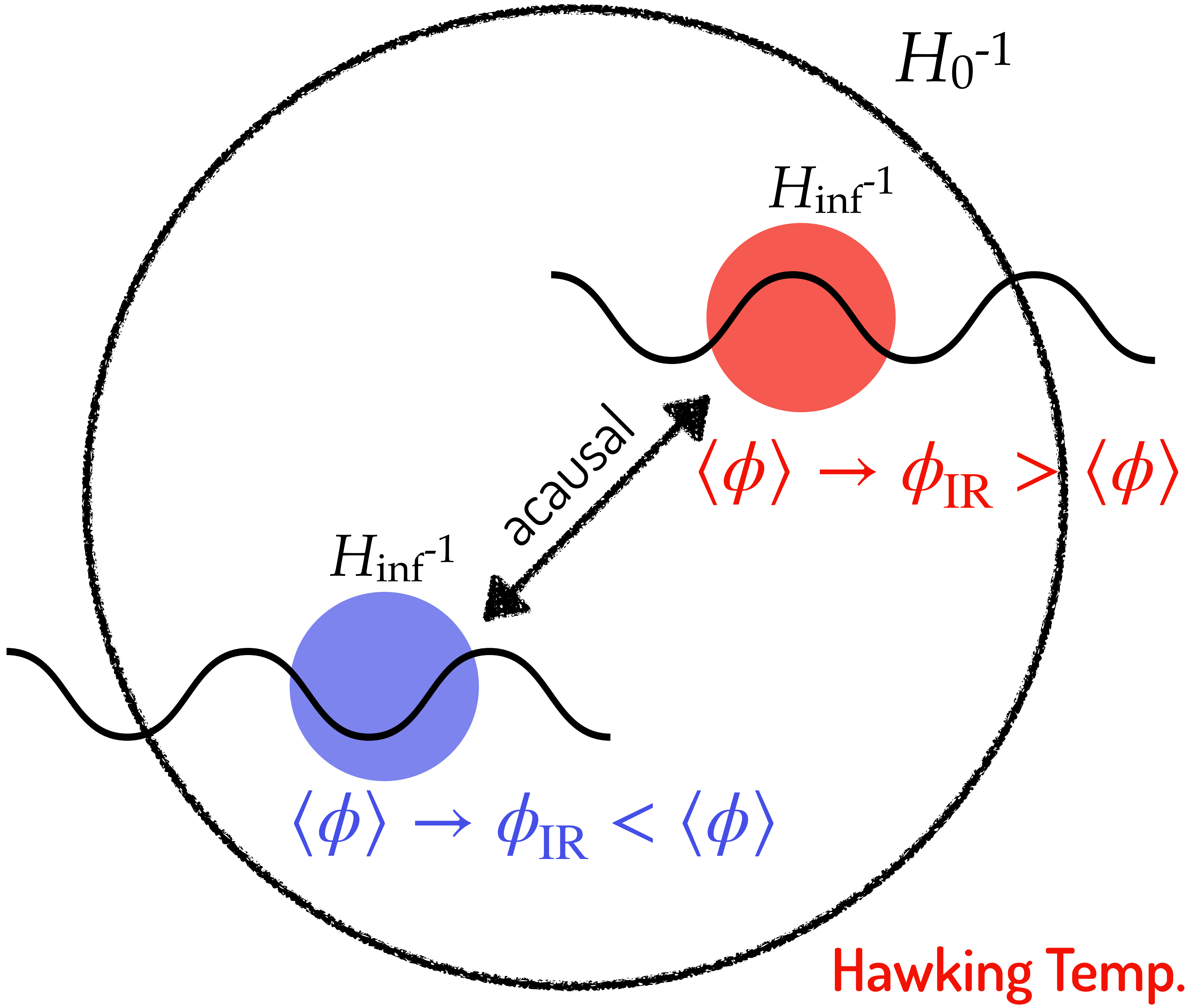
Inflaton: scalar particle

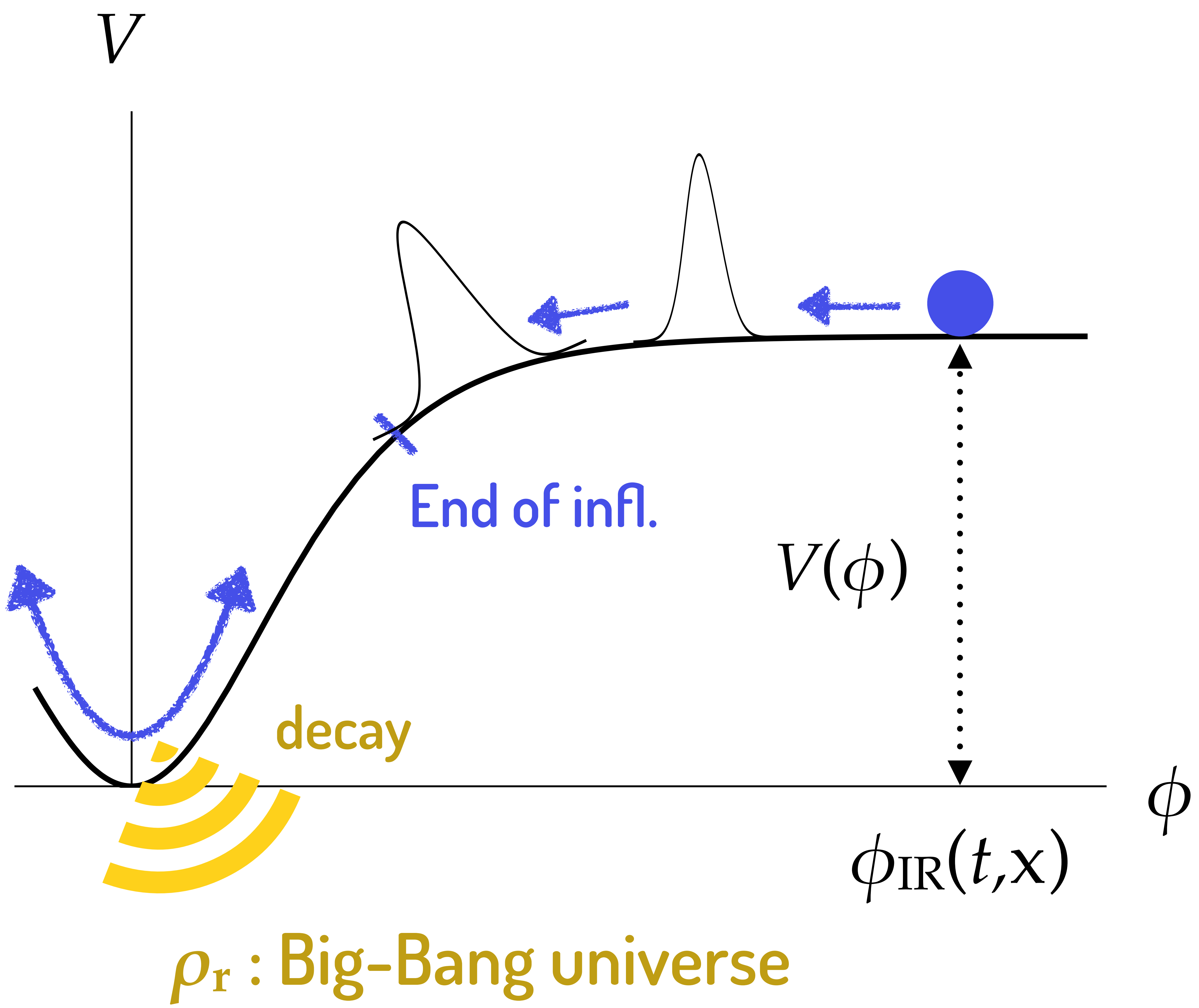


$\rho_{\rm r}$: Big-Bang universe

Stochastic Form.

Starobinsky '86





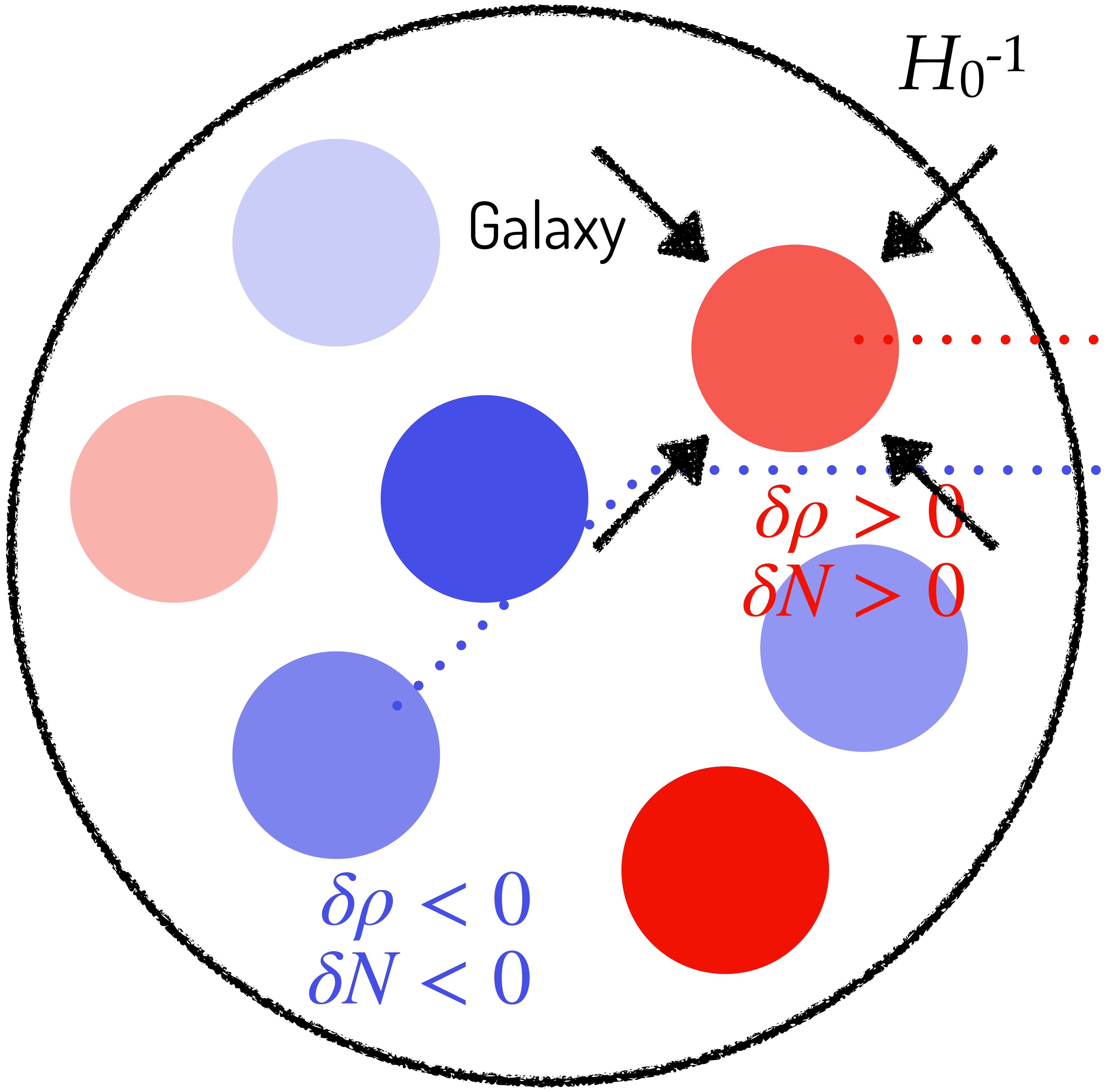
Hawking Temp.

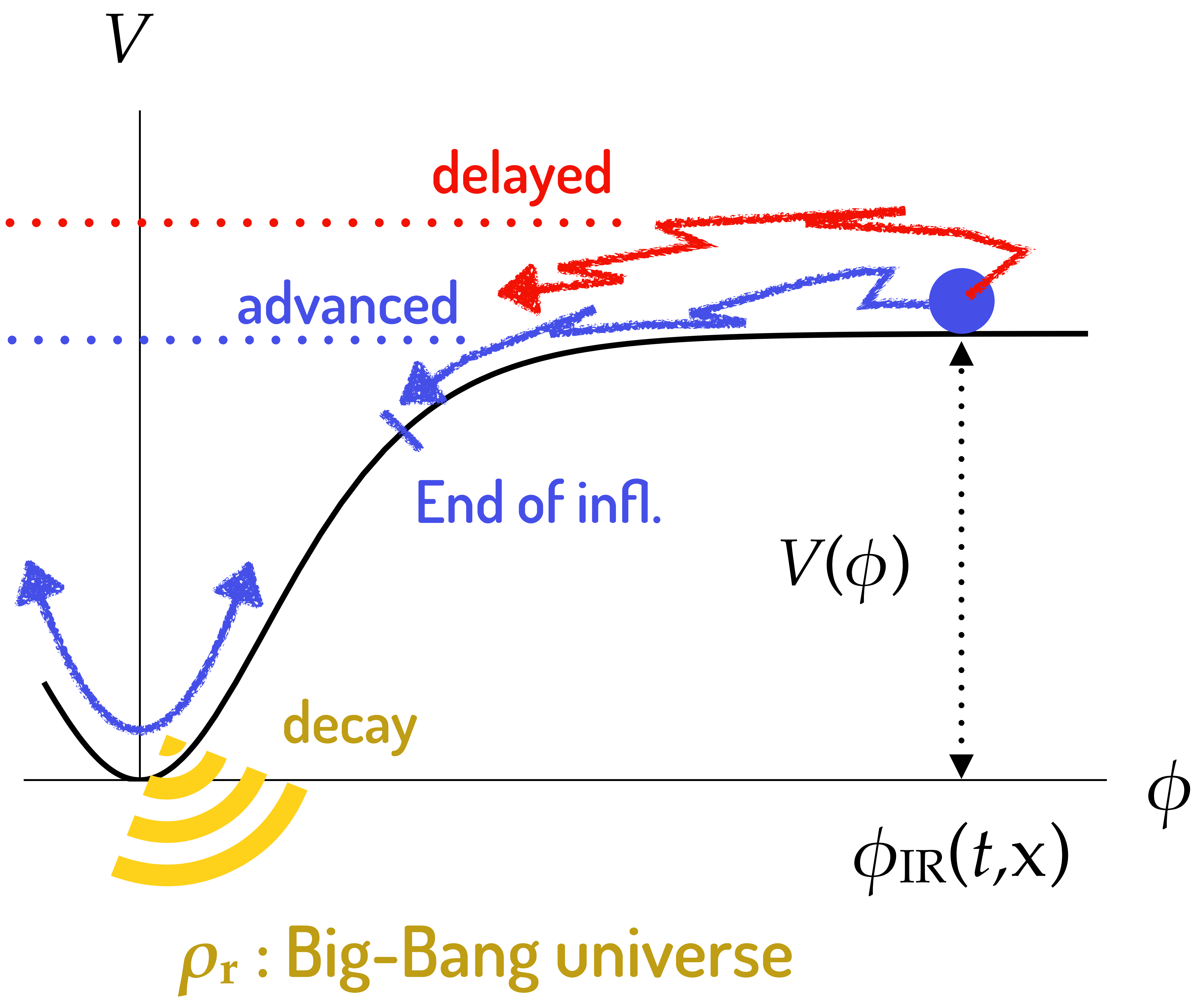
❖ Stochastic EoM: $$\frac{d\phi_{\rm IR}}{dN} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi$$

Gaussian Rand.

(conserved) δN Form.

Starobinsky '85





❖ Stochastic-δN formalism

Fujita, Kawasaki, YT, Takesato '13

Vennin & Starobinsky '15

PDE Approach

Vennin & Starobinsky '15

- Fokker-Planck eq. (diffusion)

$$\partial_N P(\phi^I;N) = -\partial_I\left[h^I P(\phi^I;N)\right] + \frac{1}{2}\partial_I\partial_J\left[A^{IJ}P(\phi^I;N)\right]$$

PDF of ϕ^I @ N

e.g. $h^I = -\frac{V^I}{3H^2}$, $A^{IJ} = \left(\frac{H}{2\pi}\right)^2\delta^{IJ}$

⇕

- adjoint FP eq.

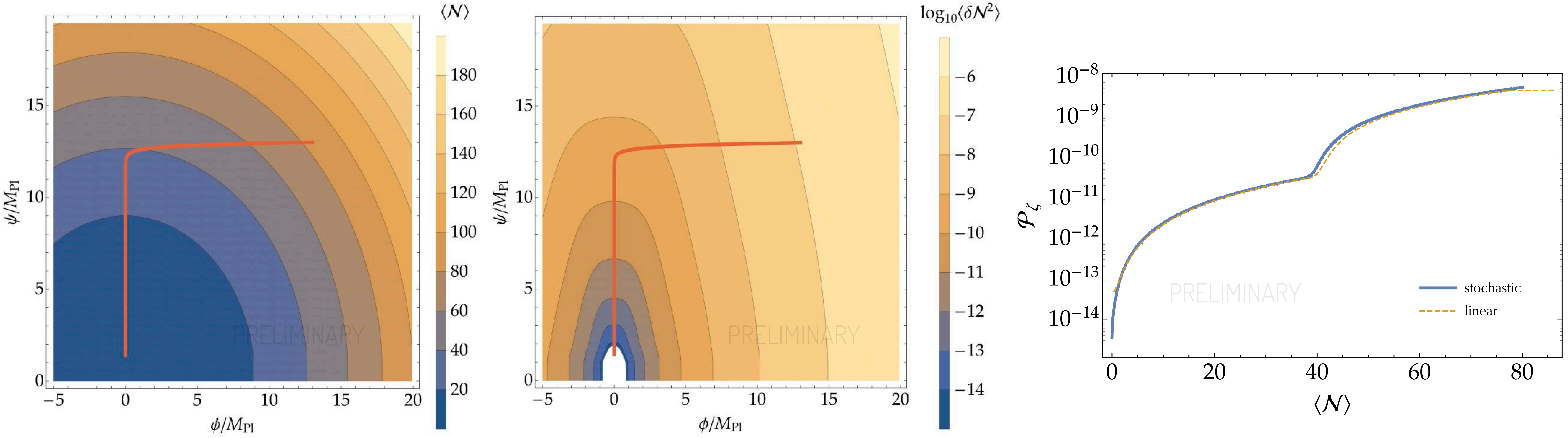
$$\partial_{\mathcal{N}}\bar{P}(\mathcal{N};\phi^I) = h^I\partial_I\bar{P}(\mathcal{N};\phi^I) + \frac{1}{2}A^{IJ}\partial_I\partial_J\bar{P}(\mathcal{N};\phi^I)$$

PDF of 1st. passage time $\mathcal{N}$ from ϕ^I

StocDeltaN.cpp

Renaux-Petel, YT, Vennin in prep.

❖ Double Mass-term $V = \frac{1}{2}M^2\phi^2 + \frac{1}{2}m^2\psi^2, \qquad M = 9m = 10^{-5}M_{\rm Pl}$

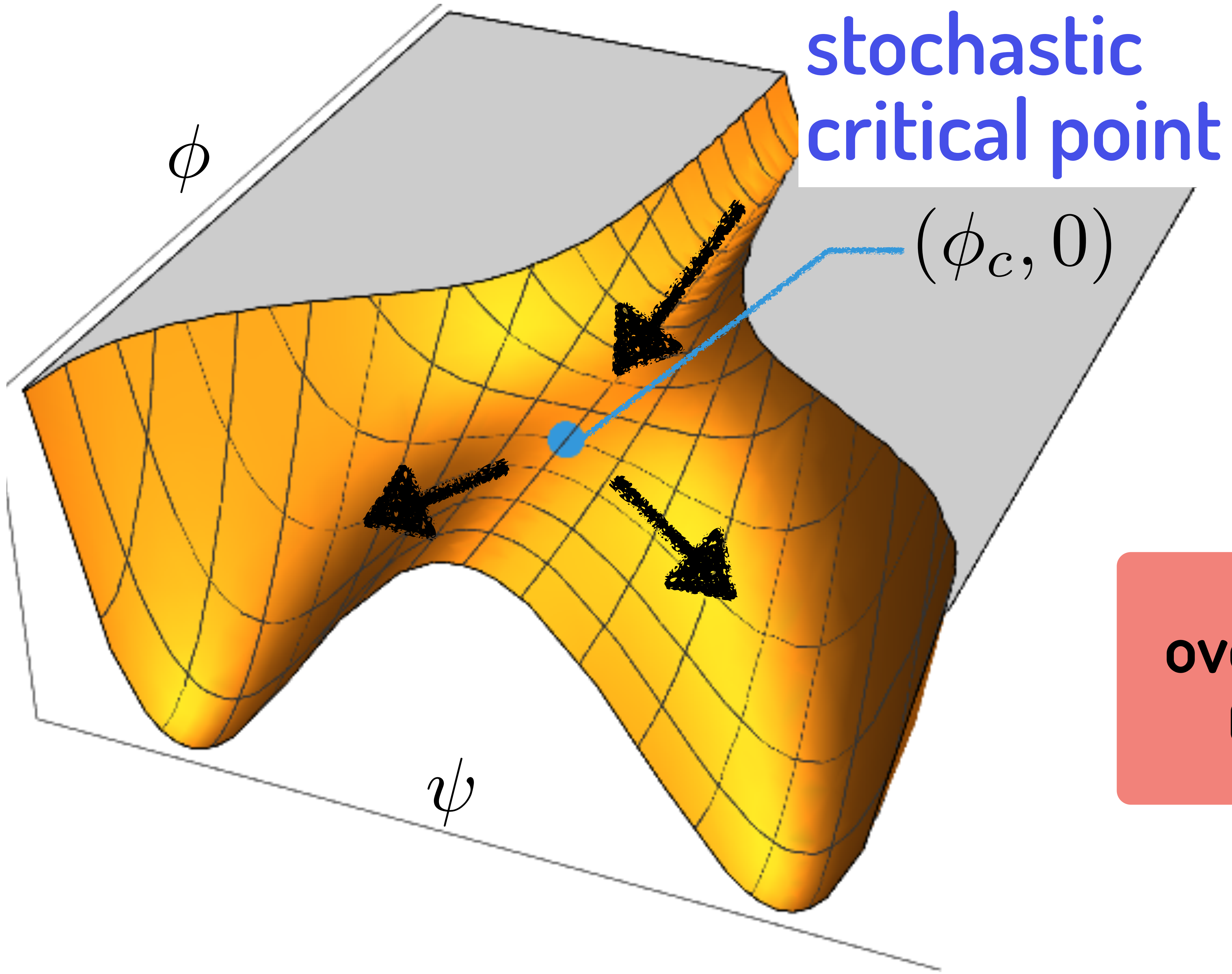


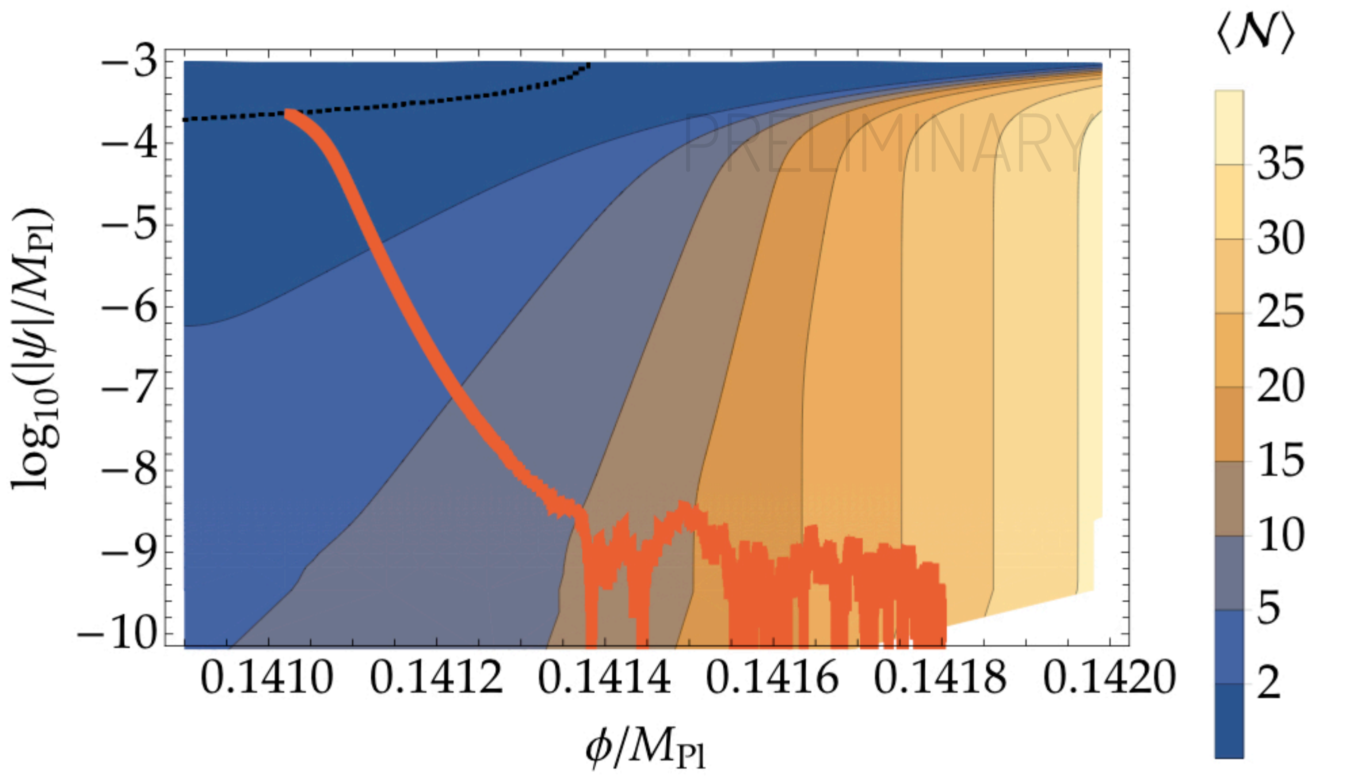
StocDeltaN.cpp

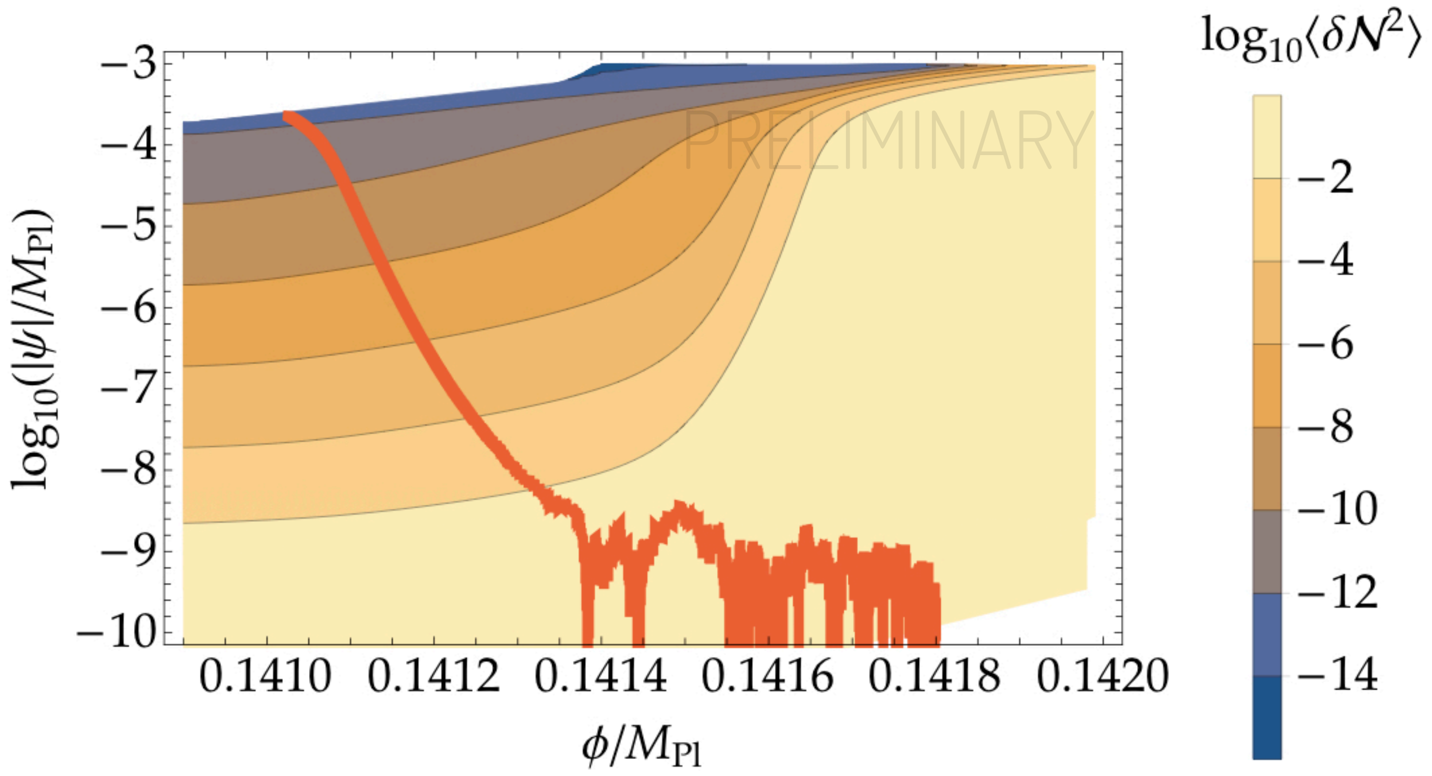
Renaux-Petel, YT, Vennin in prep.

- Hybrid Inflation

$$V = \Lambda^4 \left[\left(1 - \frac{\psi^2}{M^2}\right)^2 + 2\frac{\phi^2\psi^2}{\phi_c^2 M^2} + \frac{\phi - \phi_c}{\mu_1} - \frac{(\phi-\phi_c)^2}{\mu_2^2} \right]$$

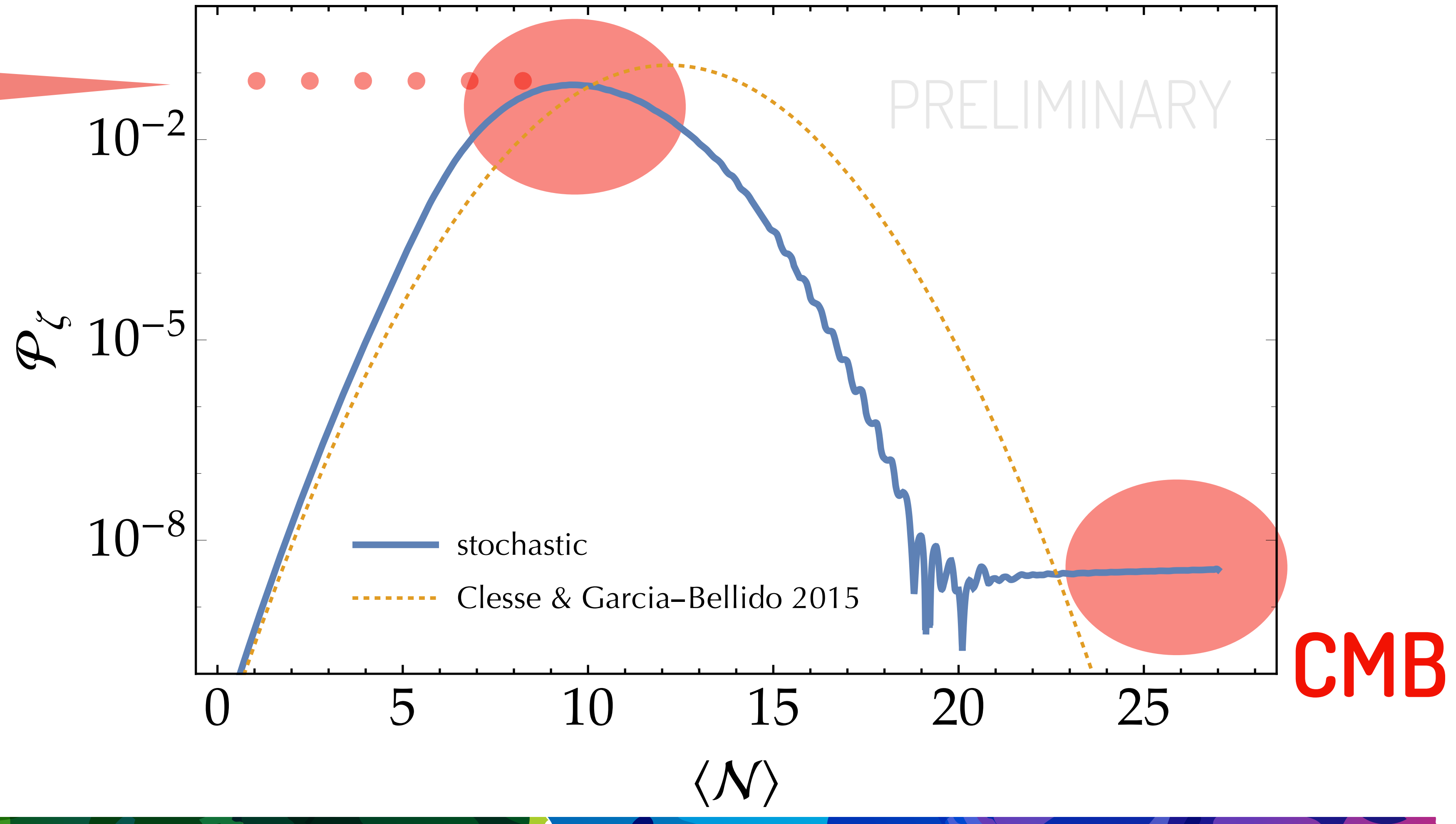






overproduce PBHs

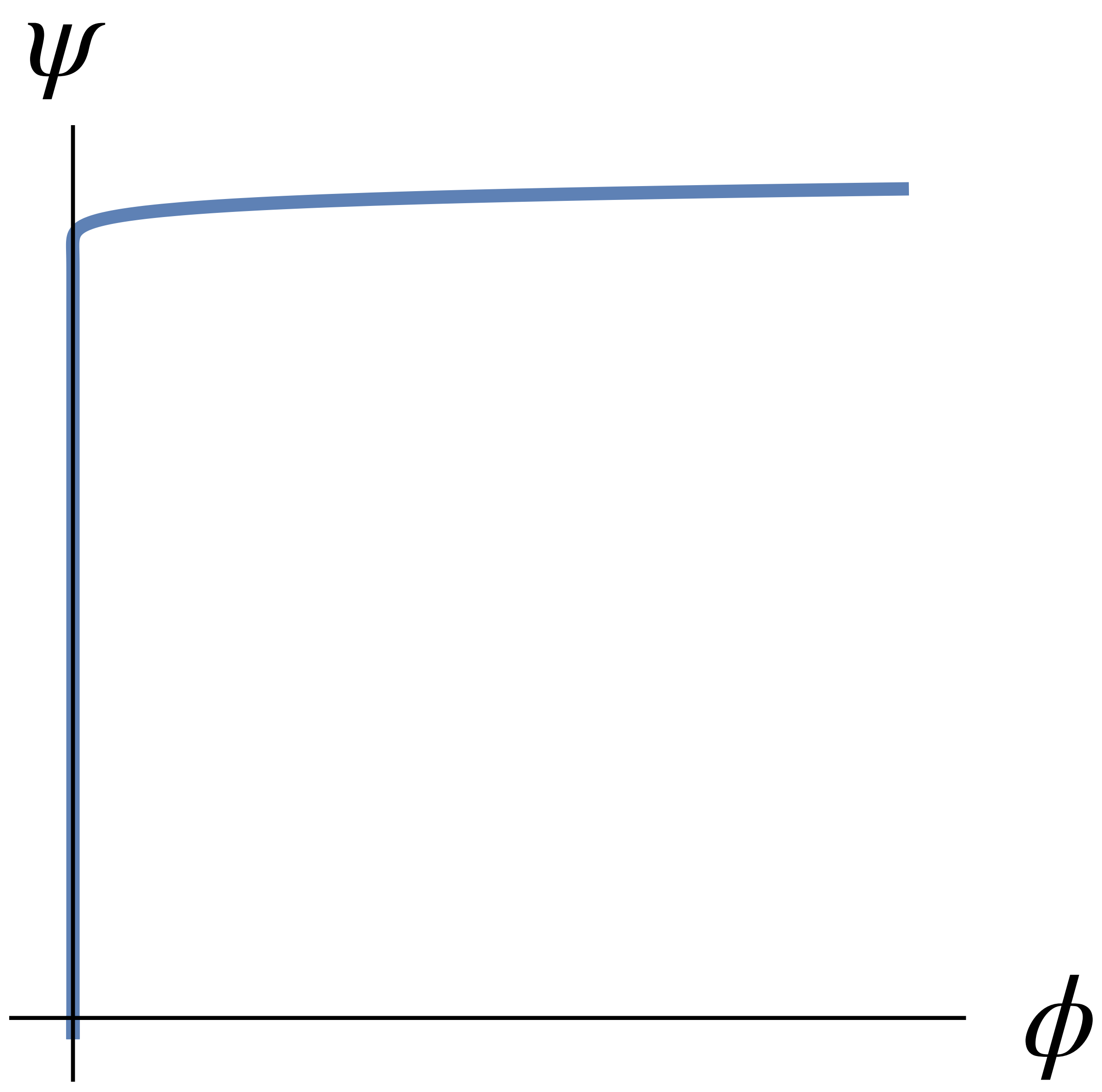
(Kawasaki, YT 2015)



GeoDesI

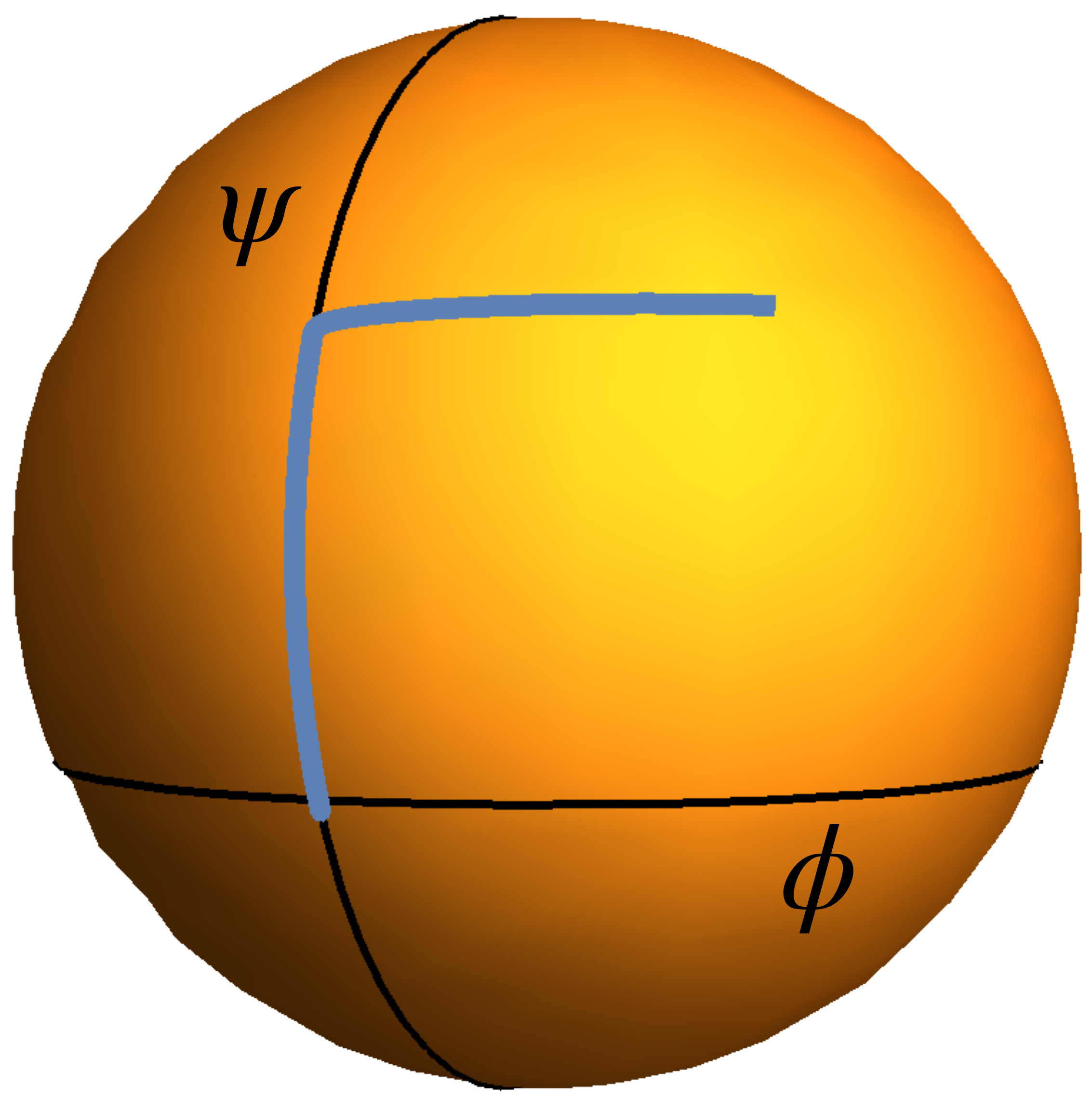
Renaux-Petel & Turzyński '15

❖ Flat Fields



$$\mathcal{L}_{\rm kin} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\psi)^2$$
$$= -\frac{1}{2}\delta_{IJ}\partial_\mu\phi^I\partial^\mu\phi^J$$

❖ Curved Fields

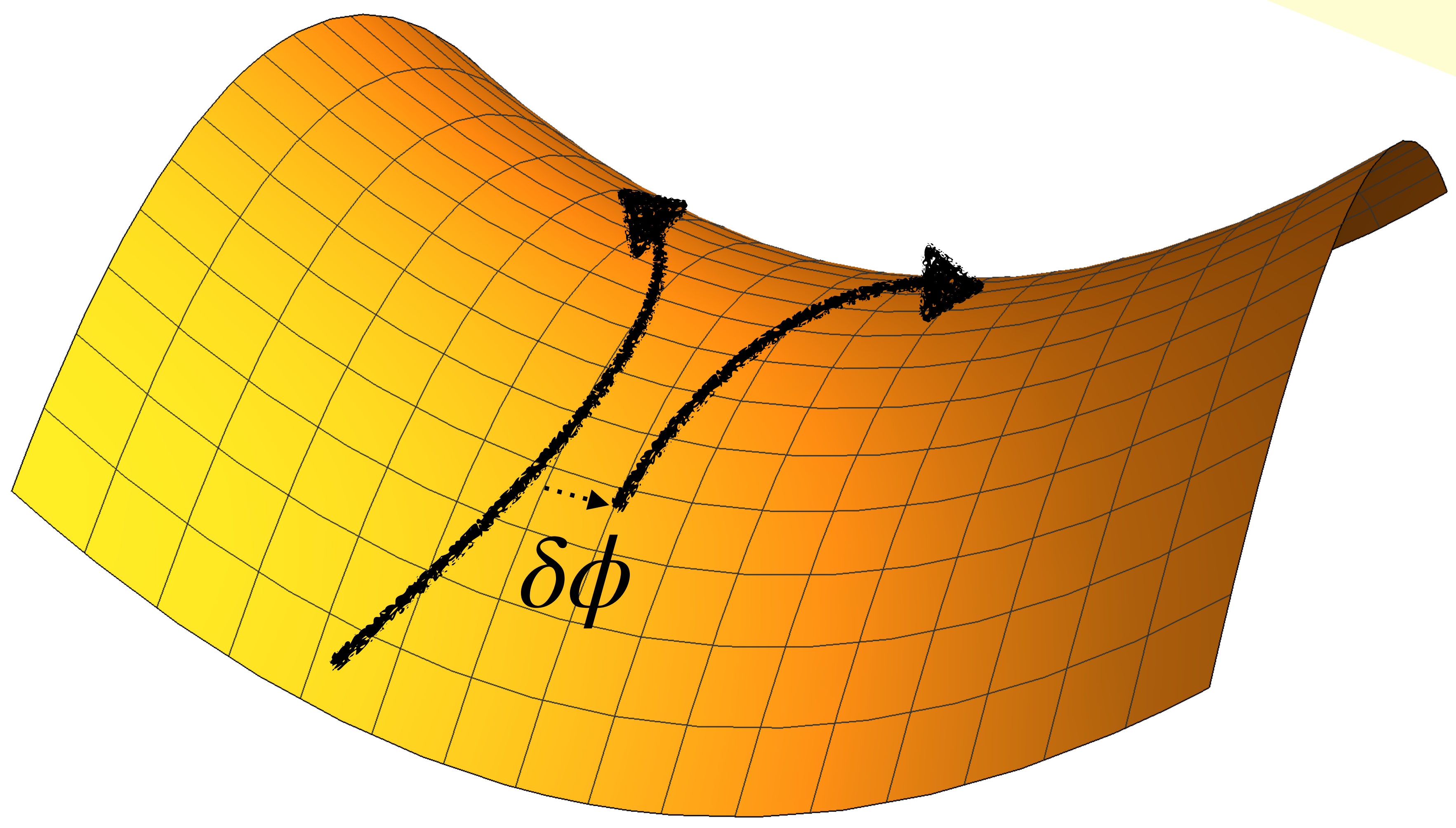


$$\mathcal{L}_{\rm kin} = -\frac{1}{2}G_{IJ}(\phi)\partial_\mu\phi^I\partial^\mu\phi^J$$

GeoDesl

Renaux-Petel & Turzyński '15

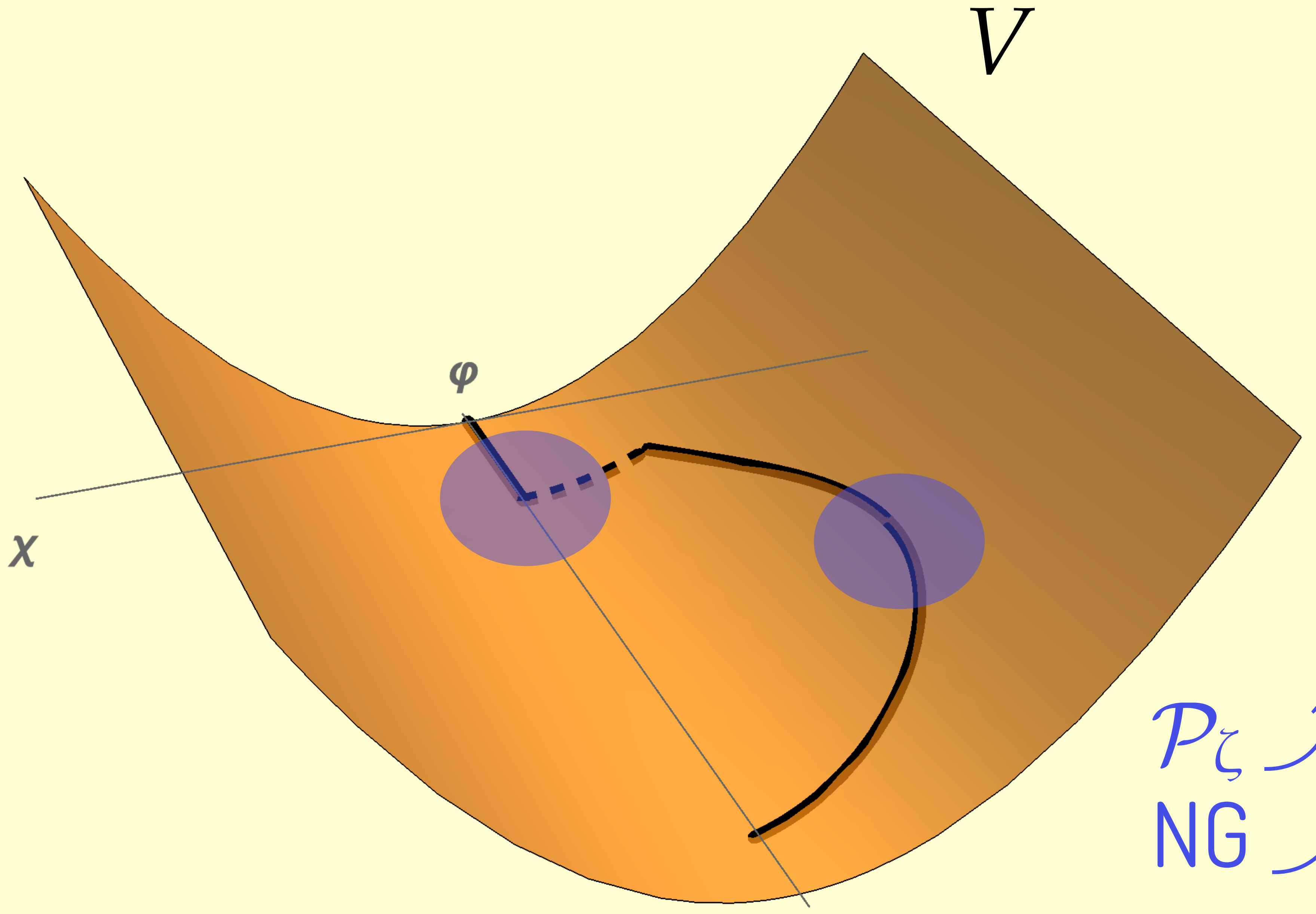
❖ Hyperbolic $R < 0$



unstable

Sidetracked Inflation

Gracia-Saenz, Renaux-Petel, Ronayne '18



$\mathcal{P}_\zeta \nearrow$

NG $\nearrow$

Generalization?

Pinol, Renaux-Petel, YT '18

Inflaton-space metric

❖ General Multi-scalar $\mathscr{L} = -\frac{1}{2} g^{\mu\nu} G_{IJ}(\phi) \partial_\mu \phi^I \partial_\nu \phi^J - V(\phi)$

Stochastic EoM ??

$$\frac{\mathrm{d}\phi^I}{\mathrm{d}N} \stackrel{?}{=} -\frac{G^{IJ}\partial_J V}{3H^2} + \xi^I, \quad \text{with } \langle \xi^I(N) \xi^J(N') \rangle = \left(\frac{H}{2\pi}\right)^2 G^{IJ} \delta(N - N')$$

Generalization?

Pinol, Renaux-Petel, YT '18

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– ~~Covariance~~ under $\phi^I \to \bar{\phi}^{\bar{I}} = \bar{\phi}^{\bar{I}}(\phi)$

$$\frac{\mathrm{d}\phi^I}{\mathrm{d}N} = -\frac{G^{IJ}\partial_J V}{3H^2} + \xi^I \quad \Rightarrow \quad \frac{\mathrm{d}\bar{\phi}^{\bar{I}}}{\mathrm{d}N} \neq -\frac{G^{\bar{I}\bar{J}}\partial_{\bar{J}} V}{3H^2} + \xi^{\bar{I}}$$

and/or

– Spurious Frame Dependence

$$\frac{\mathrm{d}\phi^I}{\mathrm{d}N} = -\frac{G^{IJ}\partial_J V}{3H^2} + \xi^I \neq -\frac{G^{IJ}\partial_J V}{3H^2} + R^I{}_{\tilde{A}} \xi^{\tilde{A}}$$

Rotation/Diagonalization

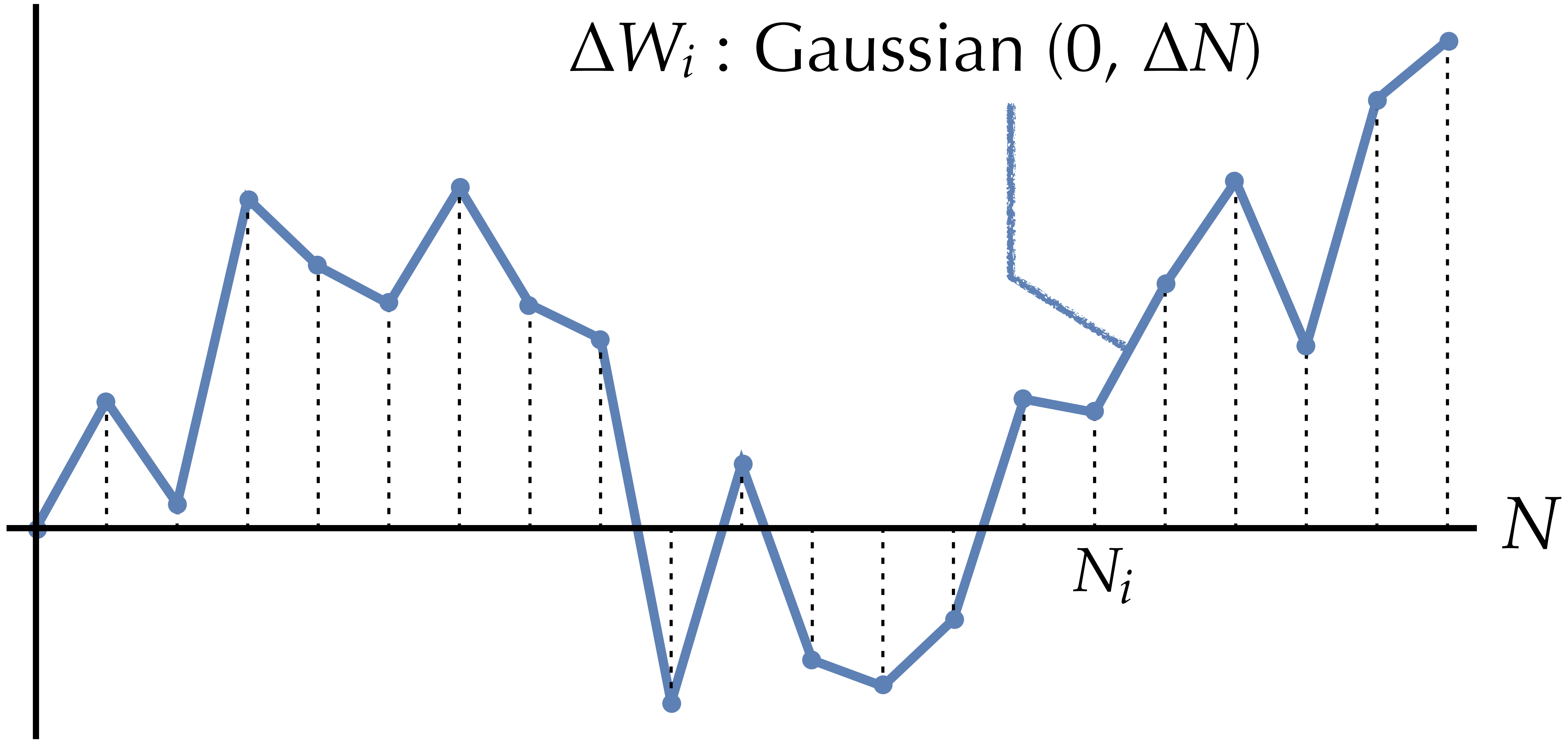
Manifestly Covariant Stochastic Inflation

Stoc. Calculus

❖ Stochastic Differential eq.

$$\frac{\mathrm{d}X(N)}{\mathrm{d}N} = A\left(X(N)\right)\xi(N), \qquad \langle \xi(N)\xi(N')\rangle = \delta(N-N')$$

Brownian $W(N)$



$$\Delta X_i = A(X_{i+\alpha})\Delta W_i$$

$$X_{i+\alpha} = (1-\alpha)X_i + \alpha X_{i+1}$$

$$0 \leq \alpha < 1$$

Itô: $X_{i+0} = X_i$

Stratonovich: $X_{i+1/2} = (X_i + X_{i+1})/2$

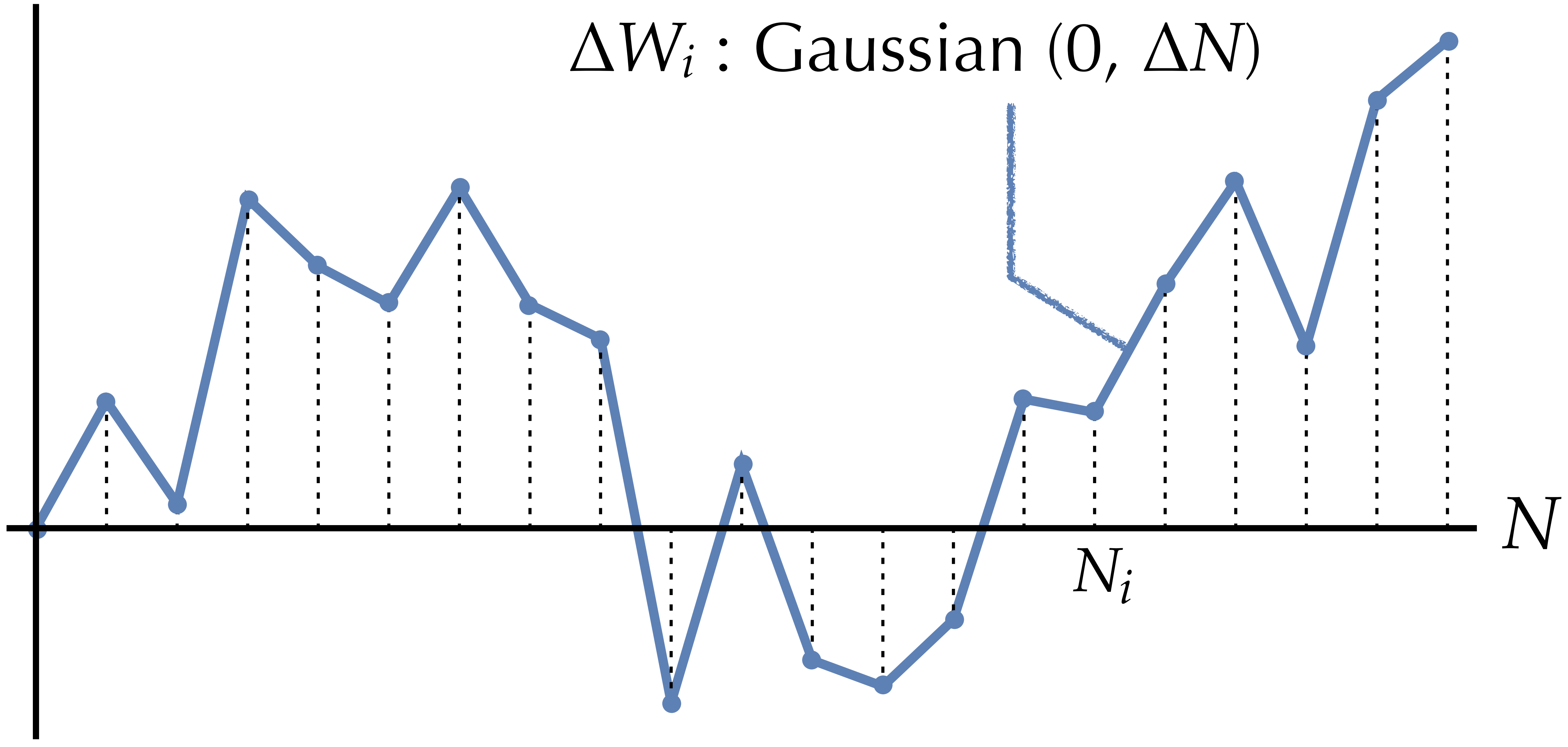
Stoc. Calculus

- Stochastic Differential eq.

$$\frac{\mathrm{d}X(N)}{\mathrm{d}N} = A\left(X(N)\right)\xi(N), \qquad \langle \xi(N$$

X's property (PDF) depends on the choice of α!

Brownian $W(N)$



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Exercise

$$\mathrm{d}Z = W\mathrm{d}W$$

❖ Itô's scheme

$$\Delta Z_i = W_i \Delta W_i \quad \Rightarrow \quad \langle \Delta Z_i \rangle = \langle W_i \Delta W_i \rangle = 0$$

No Correlation btw. current position & noise

$$Z = \int \mathrm{d}Z = \int W\mathrm{d}W = \frac{1}{2}W^2 - \frac{1}{2}N$$

$\langle Z \rangle = 0$ & $\langle W^2 \rangle = N$

- Itô's lemma: $\mathrm{d}f(N, X) = f_N \mathrm{d}N + f_X \mathrm{d}X + \frac{1}{2} f_{XX} \mathrm{d}X\mathrm{d}X$ $= A^2 \mathrm{d}N$

$$\mathrm{d}(W^2) = 2W\mathrm{d}W + \frac{1}{2} \times 2\mathrm{d}W\mathrm{d}W = 2W\mathrm{d}W + \mathrm{d}N$$

$$\Rightarrow \quad \mathrm{d}Z = \frac{1}{2}\mathrm{d}(W^2) - \frac{1}{2}\mathrm{d}N$$

Exercise

$$\mathrm{d}\bar{Z} = W \circ \mathrm{d}W$$

- Stratonovich's scheme

$$\Delta\bar{Z}_i = W_{i+1/2}\Delta W_i = \left(W_i + \frac{1}{2}\Delta W_i\right)\Delta W_i \quad \Rightarrow \quad \langle\Delta\bar{Z}_i\rangle = \frac{1}{2}\langle(\Delta W_i)^2\rangle = \frac{1}{2}\Delta N$$

$$\bar{Z} = \int \mathrm{d}\bar{Z} = \int W \circ \mathrm{d}W = \frac{1}{2}W^2 \qquad \langle\bar{Z}\rangle = \frac{1}{2}\langle W^2\rangle = \frac{1}{2}N$$

Std. Calculus !!

- Strato ⇄ Itô

$$\Delta Y_i = h(N_{i+1/2}, Y_{i+1/2})\Delta N + g(N_{i+1/2}, Y_{i+1/2})\Delta W$$

$$= h(N_i, Y_i)\Delta N + g(N_i, Y_i)\Delta W + \frac{1}{2}g_Y g\Delta N$$

Noise-induced Drift

Stoc. Anomaly

Pinol, Renaux-Petel, YT '18

$$\mathrm{d}\phi^I \overset{?}{=} -\frac{V^I}{3H^2}\mathrm{d}N + \frac{H}{2\pi}e_a^I \circ_? \mathrm{d}W^a \qquad \begin{cases} e_a^I e_a^J = G^{IJ}, \\ \mathrm{d}W^a \mathrm{d}W^b = \delta^{ab}\mathrm{d}N \end{cases}$$

- Covariance: $\phi^I \to \bar{\phi}^{\bar{I}} = \bar{\phi}^{\bar{I}}(\phi)$

  ✗ Itô's lemma

$$\mathrm{d}\bar{\phi}^{\bar{I}} = \frac{\partial \bar{\phi}^{\bar{I}}}{\partial \phi^J}\mathrm{d}\phi^J + \frac{1}{2}\frac{\partial^2 \bar{\phi}^{\bar{I}}}{\partial \phi^J \partial \phi^K}\mathrm{d}\phi^J \mathrm{d}\phi^K$$

Break Covariance

  ✓ Stratonovich

- Spurious Frame Dependence

$$\mathrm{d}\phi^I = -\frac{V^I}{3H^2}\mathrm{d}N + \frac{H}{2\pi}e_a^I \circ \mathrm{d}W^a$$

$$= -\frac{V^I}{3H^2}\mathrm{d}N + \frac{H}{2\pi}e_a^I \mathrm{d}W^a + \frac{1}{2} \times \frac{H}{2\pi}e^I_{a,J}e^J_a \mathrm{d}N$$

✗ Spurious frame dependence in Strato

$$e_a^I \to \bar{e}_{\bar{a}}^I = R_{\bar{a}}{}^b(\phi)e_b^I$$

- Stoc. vs. QFT Tokuda & Tanaka '17 & '18

$$\mathscr{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\lambda\phi^4$$

✓ $\mathcal{O}(\lambda^1) : \langle\phi^2\rangle_{\mathrm{Ito}} = \langle\phi^2\rangle_{\mathrm{QFT}}$

non-Markovian origin of noise

$$\delta\hat{\phi}^I_{\mathbf{k}} = e^I_A Q^A_a \hat{a}^a_{\mathbf{k}} + e^I_A Q^{A*}_a \hat{a}^{a\dagger}_{-\mathbf{k}}$$

$$\xrightarrow[k\ll aH]{} e^I_A Q^A_a (\hat{a}^a_{\mathbf{k}} + \hat{a}^{a\dagger}_{-\mathbf{k}})$$

($\mathrm{Im}Q^A_a \to 0$ up to const. phase)

CURRENT trs. (local frame -> global coord.) : Strato

Quantum origin of noise $\to \mathrm{d}W^a$

- Local frame mode function
- HISTORY of subhorizon dynamics : Itô

Preferred Frame

Pinol, Renaux-Petel, YT '20

Strato

$$\begin{cases} \mathrm{d}\phi^I = -\frac{V^I}{3H^2}\mathrm{d}N + e_A^I Q_a^A \circ \mathrm{d}W^a \\ \quad + \\ \mathscr{D}e_A^I = \mathrm{d}e_A^I + \Gamma^I_{JK}e_A^J \circ \mathrm{d}\phi^K = 0 \qquad (\text{or} = \Omega_A{}^B e_B^I \mathrm{d}N) \end{cases}$$

Noise Amp. in Local Frame

e_A^I is in itself a "Stoc. Variable" along the trajectory

Q_a^A is "Itô"-like

$$\mathrm{d}\phi^I = -\frac{V^I}{3H^2}\mathrm{d}N + e_A^I Q_a^A \mathrm{d}W^a + \frac{1}{2}Q_a^A \mathrm{d}e_A^I \mathrm{d}W^a = -\frac{1}{2}\Gamma^I_{JK}A^{JK}\mathrm{d}N$$

$$A^{IJ} = e_A^I Q_a^A e_B^J Q_a^B \sim \left(\frac{H}{2\pi}\right)^2 G^{IJ}$$

$$\mathfrak{D}_N\phi^I := \frac{\mathrm{d}\phi^I}{\mathrm{d}N} + \frac{1}{2}\Gamma^I_{JK}A^{JK} = -\frac{V^I}{3H^2} + \xi^I$$

Itô Covariant Derivative Graham '85

No e_A^I-dependence

Cov. Stoc. EoM

Pinol, Renaux-Petel, YT '20

take-home message

Manifestly Covariant Stochastic EoM

$$\begin{cases} \mathfrak{D}_N \phi^I = \dfrac{\pi^I}{H} + \xi^{QI} \\ \mathfrak{D}_N \pi_I = -3\pi_I - \dfrac{V_I}{H} + \xi^{PI} \end{cases}$$

w/ Itô Cov. Derivatives $\mathfrak{D}\bar{\phi}^{\bar{I}} = \dfrac{\partial \bar{\phi}^{\bar{I}}}{\partial \phi^J} \mathfrak{D}\phi^J, \quad \dfrac{\partial \bar{\phi}^{\bar{I}}}{\partial \phi^J} \mathfrak{D}\bar{\pi}_{\bar{I}} = \mathfrak{D}\pi_J$

$$\begin{cases} \mathfrak{D}\phi^I := \mathrm{d}\phi^I + \dfrac{1}{2}\Gamma^I_{JK} A^{QQJK} \mathrm{d}N \\ \mathfrak{D}\pi_I := \mathscr{D}\pi_I - \dfrac{1}{2}(\Gamma^S_{IJ,K} + \Gamma^M_{IJ}\Gamma^S_{KM})\pi_S A^{QQJK}\mathrm{d}N - \Gamma^K_{IJ} A^{QPJ}{}_K \mathrm{d}N \end{cases}$$

Conclusions

- ❖ Stochastic form. is powerfull
- ❖ Stochastic Anomaly: Itô or Strato?
  - ~~Covariance~~ or Spurious frame dependence
- ❖ Manifestly covariant SDE in Itô

Covariant FP

Appendix

covariant Itô SDE

⇔ covariant FP eq.

$D_{\phi^I} X^J = \nabla_I X^J + \Gamma^K_{IL} \pi_K \partial_{\pi_L} X^J$: (phase-space) cov. der.

phase-space PDF : scalar

∂_{π_I} : (phase-space) covariant

$$\partial_N P(\phi,\pi) = -D_{\phi^I}\left[\frac{G^{IJ}}{H}\pi_J P\right] + \partial_{\pi_I}\left[\left(3\pi_I + \frac{V_I}{H}\right)P\right]$$

$$+\frac{1}{2} D_{\phi^I} D_{\phi^J}\left(A^{\phi\phi IJ} P\right) + D_{\phi^I}\partial_{\pi_J}\left(A^{\phi\pi I}{}_J P\right) + \frac{1}{2}\partial_{\pi_I}\partial_{\pi_J}\left(A^{\pi\pi}{}_{IJ} P\right)$$